Research Statement

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1. Summary

My recent research directions center around homological mirror symmetry arising from representation theory, especially its applications in geometric Langlands with wild ramifications, and microlocal sheaves (of spectra) and its interplay with stable homotopy theory and symplectic topology. I'm also broadly interested in areas that have direct or indirect connections with symplectic geometry and representation theory. For example, my secondary research direction is in integrable and chaotic billiards.

Since this is supposed to be a short summary of my recent research directions, the structure of each of Sections 2–5 goes as follows. First, I will start with some background of the subjects and my motivations for studying them. This is by no means a comprehensive introduction of the subjects, but only to include a broader context which helps show the importance of the topics that I'm interested in. Second, I will state the main results from the relevant papers in the simplest possible way (with possibly less precision). I will then highlight the key approaches and interesting applications. The list of relevant papers is in Section 6.

2. Homological Mirror symmetry of completely integrable systems

2.1. **Background and Introduction.** One of my recent research directions is to establish homological mirror symmetry (HMS, i.e. 2d mirror symmetry) for certain classes of (algebraic) completely integrable systems that play central roles in geometric representation theory. There is a close relation to the Betti geometric Langlands (after Ben-Zvi–Nadler) with wild ramifications. Our convention in HMS is the following: starting with a (holomorphic) symplectic variety (M, ω) (usually with additional structures that we'll omit in the summary), we set it to be the A-side and set its A-model to be $\mathcal{W}(M)$—its partially wrapped Fukaya category (if well defined). Then a mirror $\check{M}$ of M (if exists) is a complex algebraic variety (or stack), referred as the B-side, satisfying $\mathcal{W}(M) \simeq \mathrm{Coh}(\check{M})$. Here all categories are dg-categories unless otherwise specified.

An important class of such integrable systems is the so called *universal centralizer* or the *Toda system* $\mathcal{J}_G$ (cf. [Lus, Kos, BFM, Gin]), which is a completely integrable system (with non-proper fibers) associated to every (connected) complex semisimple (or reductive) algebraic group G. In particular, it has a natural holomorphic symplectic structure. It has appeared in various contexts of geometric representation theory, algebraic/differential geometry (e.g. under another description as the moduli space of solutions to the Nahm equations, as in work of Atiyah-Hitchin, Donaldson, Bielawski, etc.) and mathematical physics (e.g. it is the "simplest" Coulomb branch mathematically defined by Braverman-Finkelberg-Nakajima). It was used in Ngo's proof of the fundamental lemma, and it has recently received much more attention due to the study of bi-Whittaker D-modules, which has important applications in geometric representation theory and geometric Langlands program (cf. [BZG] for a review on this).

Another important class that I'm currently focusing on is a loop group version of $\mathcal{J}_G$, which is (a partial compactification of) the *affine Toda system* (cf. [Eti]), and we'll denote it by $\mathcal{M}_G$. This completely integrable system has the great feature that it is a moduli space of Higgs bundles and it has a proper Hitchin map $f : \mathcal{M}_G \to \mathcal{A}_G$. In the world of HMS, f is a SYZ torus fibration, whose mirror is expected to be the dual torus fibration. The study of SYZ fibrations has been central in HMS conjectures. The SYZ fibrations often possess interesting singular fibers, which makes their structures quite fruitful yet causes substantial difficulties in proving HMS in higher dimensions, due to the complications of Floer cohomologies.

2.2. **Main results.** The results from (8), (11), (12), (13), (14) are reviewed below.

2.2.1. *HMS for the universal centralizers.* In papers (8) and (11), I established HMS of $\mathcal{J}_G$, for every complex reductive group G, in terms of the Langlands dual group $G^\vee$. For simplicity, I will restrict to a semisimple group G. Let $r = \mathrm{rank}\ G$, $I = \{\alpha_1, \cdots, \alpha_r\}$ a set of simple roots, $T \subset G$ be the corresponding maximal torus, T^*T be its cotangent bundle, $T^\vee$ be its dual torus, W be the Weyl group, and $Z(G)$ be the center of G. Let $\mathcal{W}(\mathcal{J}_G)$ be the partially wrapped Fukaya category of $\mathcal{J}_G$ and Coh denotes the dg-category of coherent sheaves. Then the HMS theorem in (11) states :

Theorem 1. $\mathcal{W}(\mathfrak{J}_G) \simeq \mathrm{Coh}((T^\vee)_{sc} /\!\!/ W)^{\pi_1(G^\vee)}$, where $(T^\vee)_{sc}$ means the preimage of $T^\vee$ in the simply connected form of $G^\vee$, and $/\!\!/$ means the coarse quotient.

The paper (11) not only establishes HMS, but contains many new results about the symplectic geometry/topology of $\mathfrak{J}_G$ and new techniques in calculating Floer cohomology (avoiding actual disc counting). For example, I gave a "tropical" view of $\mathfrak{J}_G$ through a map $\mathfrak{J}_G \to \mathbb{R}^r_{\geq 0}$, which results in a partial (real) compactification of $\mathfrak{J}_G$ as a Weinstein sector, and an explicit description of a Lagrangian skeleton of $\mathfrak{J}_G$ (which in particular results in a calculation of $\pi_1(\mathfrak{J}_G)$). I found Weinstein subsectors equivalent to $\mathfrak{J}_{L_J}, J \subset I$, for standard Levi subgroups $L_J \subset G$, and established functoriality among $\mathcal{W}(\mathfrak{J}_{L_J})$ (using general framework in [GPS]). A key (technical) step in the proof of the main theorem (which makes wrapping in certain calculations of wrapped Floer cohomology groups manageable) is a qualitative comparison between the restriction of the integrable system on $\mathfrak{J}_G$ and the standard one on the "subsector" T^*T, which are essentially different but become "close" to each other near the sector boundary of $\mathfrak{J}_G$. Lastly, let me comment that Theorem 1 should be viewed as the decategorification of the conjectural 3d mirror symmetry (or 2-categorical Koszul duality) between $\mathfrak{J}_G$ and $T^*(BG^\vee)$ (cf. [Hil] for a discussion on this, and [GaHi] for the hypertoric case).

2.2.2. *HMS for the affine Toda systems.* In (12), a joint work with Zhiwei Yun (which will be available very soon), we give the description of $\mathcal{M}_G$ as a moduli space of Higgs bundles on $\mathbb{P}^1$ with certain level structures at $0, \infty$. This enables us to formulate the mirror conjecture on $\mathcal{M}_G$, which is an instance of Betti Geometric Langlands duality with wild ramifications.

Conjecture. Let G be any complex semisimple group of adjoint form. Then the mirror of $\mathcal{M}_G^\circ$ (the neutral component of $\mathcal{M}_G$) is the group-group universal centralizer $\mathfrak{J}_{G^\vee}^{G^\vee_{\mathrm{ad}}}$. That is, $\mathcal{W}(\mathcal{M}_G^\circ) \simeq \mathrm{Coh}(\mathfrak{J}_{G^\vee}^{G^\vee_{\mathrm{ad}}})$, where $G^\vee_{\mathrm{ad}}$ is the adjoint form of $G^\vee$.

One of the main results of (12) is establishing the mirror conjecture.

Theorem 2. The above mirror conjecture holds for all G. There are also natural extensions when G is (connected) reductive.

Our main steps of the proof go as follows. We cover $\mathcal{M}_G^\circ$ by universal centralizers $\mathfrak{J}_{L_J}, J \subsetneq \widetilde{I}$, associated to standard pseudo-Levi L_J of G, where $\widetilde{I} = I \cup \{\alpha_0\}$ consists of affine simple roots. We endow $\mathcal{M}_G^\circ$ with a Weinstein manifold structure [1] and realize $\mathfrak{J}_{L_J}$ as "Weinstein subsectors", so that we can compute $\mathcal{W}(\mathcal{M}_G^\circ)$ as a colimit of $\mathcal{W}(\mathfrak{J}_{L_J}), J \subsetneq \widetilde{I}$. Then the mirror B-model can be computed using Theorem 1 together with the functorialities established in (11). The last step (also highly non-trivial) is a comparison of the result with $\mathrm{Coh}(\mathfrak{J}_{G^\vee}^{G^\vee_{\mathrm{ad}}})$, which uses a few crucial properties of representation theory of (pseudo-)reflection groups.

A remarkable feature that I'm especially excited about is that though the non-properness of the integrable system $\mathfrak{J}_G$ puts itself outside the SYZ picture, its HMS has nevertheless beautiful applications in the HMS of the SYZ fibration $\mathcal{M}_G^\circ \to \mathcal{A}_G$, which bypasses all the complications caused by singularities of the fibers from the traditional perspective. I expect this approach will have broader and substantial applications in HMS in the future.

Lastly, let me briefly mention some future directions along this line. First, there is conjecturally a Simpson correspondence between $\mathfrak{J}_{G^\vee}^{G^\vee_{\mathrm{ad}}}$ and $\mathcal{M}^\circ_{G^\vee_{\mathrm{ad}}}$, and a P=W conjecture. Currently, we are able to establish a homotopy equivalence between the two spaces. Second, as $\mathcal{M}_G^\circ$ is a loop group version of $\mathfrak{J}_G$, one can ask if there is an elliptic version of the space as well as HMS. For now, we have a candidate for this, but HMS is under investigation.

2.2.3. *Cohomology of $\mathfrak{J}_G$, and cluster structure on the group-group universal centralizer.* In (14), which is work in progress, I'm studying the rational cohomology of the universal centralizer $\mathfrak{J}_G$. The main result is the following (using the notations from §2.2.1)

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[1] This is not automatic since there is no natural $\mathbb{C}^\times$ or $\mathbb{R}_+$-action on $\mathcal{M}_G^\circ$. On the other hand, a hyperkahler rotated (not the present) Kähler structure (as a wild character variety) would give a desired Weinstein structure, but the hyperkahler structure on $\mathcal{M}_G^\circ$ is only conjectural.

Theorem 3. Let G be semisimple of rank r. Then the cohomology $H^*(\mathcal{J}_G;\mathbb{Q})$ has a pure Hodge structure. In particular, the Poincaré polynomial of $\mathcal{J}_G$ is given by

$$P_{\mathcal{J}_G}(t)=\sum_{J\subset I}|\pi_0(Z(L_J))|t^{2|J|}(1-t^2)^{r-|J|}.$$

The proof is based on the Weinstein handle decomposition obtained in (11). This result and techniques will be applied to subsequent work of (12), e.g. showing pure Hodge structure on $H^*(\mathcal{M}_G^\circ;\mathbb{Q})$.

In (13), joint work in progress with Ben Webster, we prove that $\mathcal{J}_G^{G_{\mathrm{ad}}}$ (assuming G is semisimple and simply connected) is a cluster variety, confirming a conjecture of Cautis-Williams [CaWi] for the non-quantized version. The proof is based on a more refined (and more difficult) "Bruhat decomposition" of $\mathcal{J}_G^{G_{\mathrm{ad}}}$ than for $\mathcal{J}_G$ as in (11).

3. Microlocal sheaves and symplectic topology

3.1. **Background.** Another major component of my recent research is the study of microlocal sheaf categories on (exact) symplectic manifolds and applications to symplectic topology, representation theory and HMS. Microlocal sheaf theory was developed by Kashiwara–Schapira in the 90s. It has found numerous remarkable applications in symplectic topology after the work of Nadler–Zaslow [NaZa, Nad1], and Tamarkin [Tam1]. The work of Nadler and Zaslow establishes an equivalence between the category of constructible sheaves on a closed manifold X, denoted by $\mathrm{Shv}_c(X)$, and the (infinitesimally wrapped) Fukaya category of T^*X, denoted by $\mathrm{Fuk}(T^*X)$. It has led to a sequence of interesting applications in HMS, cf. [FLTZ, Kuw, Nad2] and many more. The work of Tamarkin [Tam1] gives a microlocal condition for non-displaceability of two sets in a cotangent bundle T^*X. It opened up a new way to study questions in symplectic topology using microlocal techniques, cf. [GKS, Guil, Chiu, Tam2] and many others. The key structure is a sheaf of microlocal categories $\mu\mathrm{Shv}_{L;\mathbf{k}}$ on a Lagrangian subvariety $L\subset T^*X$ (more generally a co-isotropic subvariety), sometimes referred as the Kashiwara–Schapira stack, which is invariant under Hamiltonian isotopies. If the supporting Lagrangian is smooth, then the Kashiwara–Schapira stack is a locally constant sheaf of categories, with fiber equivalent to the category of $\mathbf{k}$-modules, denoted by $\mathrm{Mod}(\mathbf{k})$, for any given coefficient ring $\mathbf{k}$.

3.2. **Main results.** The papers (4), (5), (7) are reviewed below.

3.2.1. *Brane structures in microlocal sheaves.* In (4), a joint work with David Treumann, we generalized the construction of the Kashiwara–Schapira stack for any commutative ring spectrum $\mathbf{k}$, leading to interesting connections to stable homotopy theory. Our motivation was to give an alternative construction of Nadler-Zaslow's functor, from Lagrangian branes to sheaves, without using Floer theory. This has the benefit of direct generalizations to ring spectra coefficients. For a Legendrian $\Lambda\subset T^\infty X$ (the infinity contact boundary of T^*X), let $\mathrm{Shv}_\Lambda(X;\mathbf{k})$ denote the presentable stable ∞-category of sheaves on X with singular support in Λ.

Theorem 4 (simplified version). For any smooth exact Lagrangian $L\subset T^*X$ with a conical end to a smooth Legendrian Λ (possibly empty), there is a fully faithful functor $\Gamma(L,\mu\mathrm{Shv}_{L;\mathbf{k}})\hookrightarrow\mathrm{Shv}_\Lambda(X;\mathbf{k})$.

Here we refer to $\mu\mathrm{Shv}_{L;\mathbf{k}}$ as the *sheaf of brane structures on* L, and a Lagrangian *brane*, in our more general sense, is a pair $(L,\rho\in\Gamma(L,\mu\mathrm{Shv}_{L;\mathbf{k}}))$. The theorem then gives a desired Nadler-Zaslow functor. The proof of the theorem is built on Guillermou's "doubling trick" for quantizing compact exact Lagrangians. Another notable contribution of our work is the establishment of several main results in microlocal sheaf theory in the presentable setting. Our approach (inspired by Tamarkin) is based on "Ω-lenses", "standard" contact transformations and Guillermou-Kashiwara-Schapira's quantization of Hamiltonian isotopies, which I expect to have broader applications in future development of microlocal sheaves of spectra.

3.2.2. *Sheaves of spectra and the J-homomorphism.* In papers (5) and (7), I resolved the following fundamental question in microlocal sheaves of spectra:

Question: It is standard that $\mu\mathrm{Shv}_{L;\mathbf{k}}$ on a smooth Lagrangian L, as a locally constant sheaf of categories with fiber $\mathrm{Mod}(\mathbf{k})$, is classified by a map $L\to B\mathrm{Pic}(\mathbf{k})$ up to homotopy. What is this classifying map? Is there a symplectic topological description of it?

Here $B\mathrm{Pic}(\mathbf{k})$ is the classifying space of the $\mathbf{k}$-linear automorphism group of $\mathrm{Mod}(\mathbf{k})$. In (7), I proved the following theorem (whose statement was inspired by the work [AbKr] and was expected in (4)):

Theorem 5 (rough form). The classifying map of $\mu\mathrm{Shv}_{L;\mathbf{k}}$ factors through the stable Gauss map $\gamma : L \to U/O$ and the delooping of the J-homomorphism $BJ : U/O \to B\mathrm{Pic}(\mathbf{S})$, where U/O is the stable Lagrangian Grassmannian and $\mathbf{S}$ is the sphere spectrum.

A key step of the proof is a sheaf quantization result of a (stable) Hamiltonian $G := \coprod_n BO(n)$-action on $\varinjlim_N T^*\mathbb{R}^N$ carried out in (5), which builds a natural bridge from sheaves of spectra to the J-homomorphism. Let $VG \to G$ be the tautological vector bundle on G. The quantization result can be roughly stated as follows.

Theorem 6. There is a (stable) correspondence $\widehat{Q} \xleftarrow{\psi} VG \xrightarrow{\pi} Q^0$ of topological spaces, naturally associated to the Hamiltonian G-action, such that $\psi_*\pi^!$ induces a canonical equivalence $\mathrm{Loc}(Q^0;\mathbf{S}) \xrightarrow{\sim} \mathrm{Loc}(\widehat{Q};\mathbf{S})^J$. Here Q^0 is contractible, $\widehat{Q}$ is topologically a G-torsor, $\mathbf{S}$ is the sphere spectrum, and $\mathrm{Loc}(\widehat{Q};\mathbf{S})^J$ is the ∞-category of J-equivariant local systems on $\widehat{Q}$.

Since the main results (Theorem 5 and 6) both involve infinite levels of homotopy coherent data in some ∞-categories, one must (as always in higher categories) choose some models or machineries so that these data can be encoded instead by some "discrete" data, through suitable functorialities. A great portion of the work (5) was to produce some general construction/framework to achieve such goals. For example, we used the $(\infty, 2)$-category of correspondences by Gaitsgory-Rozenblyum to deal with functoriality issues on sheaves of spectra. We produced several new constructions of commutative algebra objects and (right-lax) homomorphisms in a symmetric monoidal $(\infty, 2)$-category of correspondences, out of "discrete" data. We applied these results to achieve functorialities and symmetric monoidal structures involved in the main theorems, which we expect to have future interesting applications. Moreover, we gave an alternative realization of the Thom construction, hence the J-homomorphism, using correspondences.

3.2.3. *Applications.* Let me briefly mention a few important applications of the above theorems. First, in (5), we gave an enrichment of the Main Theorem of the stratified Morse theory by Goresky-MacPherson, regarding the monodromy of taking microlocal stalks at a given covector over the space of stratified Morse functions (under stabilization), by the J-homomorphism. Second, in (7) we proved the triviality of the sheaf of brane structures (over the sphere spectrum) on any compact exact Lagrangian submanifold in a cotangent bundle (using sheaf-theoretic methods), recovering the main result in [AbKr], which is an implication of the long standing Arnold's nearby Lagrangian conjecture. Third, Theorem 5 is important in developing and calculating microlocal sheaf categories on a Weinstein manifolds, as defined by Nadler-Shende.

4. Billiards: A class of discrete Hamiltonian dynamics

4.1. **Background and motivation.** A (discrete) dynamical system is just an automorphism F of a space X, depending on which category we are in, e.g. smooth, algebraic, symplectic, etc., and people are studying the long term behavior of iterations of F. Planar convex billiards, associated to convex domains $Q \subset \mathbb{R}^2$ with smooth boundaries, form one of the most classic and concrete sources of Hamiltonian dynamics (on the other hand, notoriously difficult). Let S^*Q denote the cosphere bundle on Q. Then the phase space of the billiard is just an open "half" of $S^*Q|_{\partial Q}$ (which has a natural exact symplectic structure). There is an intimate relation of the dynamics with geodesic flow with respect to the Euclidean geometry, under the name *billiard flow.*

Given a dynamical system, there are two major questions that people are interested in: i) is the system integrable? ii) if not, how chaotic is it? Integrability for billiards means the phase space (as a 2d symplectic manifold) is foliated by Lagrangian curves invariant under the billiard map. These invariant curves are represented by caustics on the configuration space $\mathbb{R}^2$. A long standing conjecture of Birkhoff (in one version) asserts that the only integrable smooth planar billiards are the ellipses. There have been progresses in proving various restricted versions of the conjecture, e.g. a polynomial version of Birkhoff's conjecture is established in [Glu] based on algebraic geometry, but the general version remains open.

More recently, there has been great interest into billiards with *singular* boundaries. For example, a class that was studied in my recent research is the so called (asymmetric) lemon billiard. It is given by the intersection of two round discs of radii r and R respectively (with $r \le R$), with their centers having distance

b, denoted by $Q(r, b, R)$. "Asymmetric" refers to the case that $r < R$. There are mainly two reasons that such billiards are interesting objects to study. First, it is well known that if Q is convex and has smooth boundary, then there exists a Cantor set of invariant curves near the boundary of the phase space. So to seek for dynamics that are ergodic or possess certain strong chaotic properties like hyperbolicity, one must extend the class of smooth convex billiards to singular ones. Second, reflections on circular arcs are easy to understand, and numerical simulations demonstrate that the phase portraits of asymmetric lemon billiards range from "integrable" ones to completely chaotic ones.

4.2. **Main results.** The papers (9), (10), (6) reviewed below are joint work with Pengfei Zhang.

4.2.1. *Birkhoff normal form and twist coefficients of periodic orbits of billiards.* Given a Hamiltonian system (discrete or continuous), the Birkhoff normal form is an important tool to understand the local dynamics of the system around an elliptic periodic orbit. The system is analytically integrable around an elliptic periodic orbit only if it has a convergent Birkhoff normalization. For 2d systems, the twist coefficients (also known as Birkhoff coefficients) play an important role in the stability properties of the periodic orbits.

In (9), we computed the second twist coefficients of the 2-periodic orbits for various billiards and obtained an explicit formula, expressed as a rational function (up to certain square roots from eigenvalues of the tangent map along the periodic orbits) with integer coefficients in terms of the geometric quantities of the underlying table, including the curvatures and their derivatives with respect to the arc length parameters. The formulas are very nice, just a little bit long to be reproduced here. For more details, see (9). Let me comment on the main challenge of this project. Due to the huge amount of computations, we created a Mathematica notebook (available online) to do the calculations of the twist coefficient. However, the general expression obtained by Mathematica was too long (far from reasonably simplified) to provide any insightful information. To resolve this, we first did a specialization to the easier cases of symmetric billiards and obtained an explicit formula for the twist coefficient. Then we utilized several algebraic tools and properties of the twist coefficient to formulate an ansatz about the structure of the function in the general case, reducing it to the problem of undetermined coefficients of multivariate polynomials.

4.2.2. *Homoclinic intersections of lemon billiards.* Let $Q(b) := Q(1, b, 1)$ be a lemon table. Numerical results suggested that the lemon billiards have positive entropy for all $0 < b < 2$. However, verifying such claims is known to be very difficult for systems with elliptic regions. The phase portrait (coming from drawing many orbits in different colors, under some standard coordinates) in Fig. 1 (left picture) has a beautiful feature (partly due to its symmetry) that there is a chain of secondary elliptic islands seemingly enclosed by the stable and unstable manifolds of the hyperbolic periodic orbits of period 6 (the blue and red orbits in Fig. 1, bifurcating from the 2-periodic orbit as b increases from 1.5), all of which happen in the main island around the 2-periodic orbit of the lemon billiard. My initial motivation was to investigate whether these stable and unstable manifolds could form heteroclinic connections. If they could not, then the symmetries of the lemon billiards could be used to show the existence of crossing heteroclinic and homoclinic intersections and hence (by general theory) positive topological entropy. Still this is a very difficult question in general due to its highly nonlinear nature.

In (10), we obtained the following result:

Theorem 7. There exists $\delta > 0$ such that for almost every $b \in (1.5, 1.5 + \delta)$, the hyperbolic periodic-6-orbits of the lemon billiard on $Q(b)$ admit topological crossing homoclinic intersections (so no heteroclinic connections). In particular, such billiards have positive topological entropy.

Let me briefly highlight some of the key approaches, motivated from representation theory and contact geometry, in the proof of the theorem that are "untraditional" in the study of billiards. At the "critical" parameter $b = 1.5$, the 2-periodic orbit $\mathcal{O}_2$ of the lemon billiard is at resonance (but is nevertheless stable thanks to the positive twist coefficient). We introduced a new coordinate system measuring the oriented distances of each trajectory to the two centers of the arcs. Then the tangent map at the two sides of the billiard table along the orbit $\mathcal{O}_2$, generates the reflection representation[2] of S_3 on $\mathbb{R}^2$. This means that near that orbit and for b close to 1.5, the dynamics is a small perturbation of this integrable linearized one.

[2] There is a countable set of b where one can get other dihedral groups acting on the plane. See Section 6 in our paper (10) for more details on why each map is anti-symplectic.

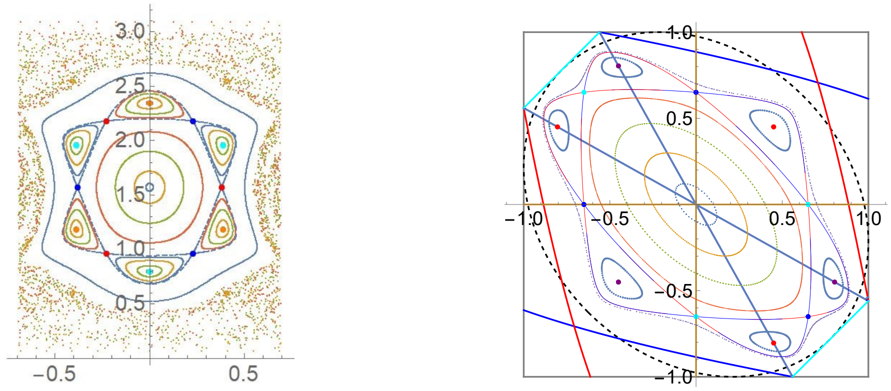

FIGURE 1. Left: Part of the phase portrait of the lemon billiard corresponding to one arc with $b = 1.54$. Right: Phase portrait using our new coordinate system with $b = 1.56$.

The group S_3 (now viewed alternatively from the configuration space) also plays an important role in our classification of all possible patterns of potential 6-periodic orbits. This allowed us to conclude that there are only two such bifurcating periodic orbits (up to symmetry), one hyperbolic and one elliptic. See the figure on the right side of Fig. 1. In our new coordinate system, the billiard reflections on the two sides of the lemon table become *nonlinear* horizontal and vertical reflections on the new phase space about their respective reflection axes (i.e. fixed locus), which are the two slant lines through the center in Fig. 1. Properties of these reflections and additional symmetries play a crucial role in our study of behaviors of the stable and unstable manifolds of the bifurcated hyperbolic periodic-6-orbits.

4.2.3. *Hyperbolicity of asymmetric lemon tables.* At the other extreme, as opposed to lemon billiards that always admit elliptic islands, the asymmetric lemon billiards admit ones that are completely hyperbolic, which is a strong chaotic property. It is conjectured in [BZZ] that asymmetric lemon billiards are hyperbolic under the “major-arc” condition: $b^2 + r^2 < R^2 < (b + r)^2$. In (6), we were able to prove a weaker version of the conjecture:

Theorem 8. (Simplified version) Assume $b^2 + r^2 < R^2 < (b + r)^2$ and $r \ll R$. Then the asymmetric lemon billiard $Q(r, b, R)$ is hyperbolic.

5. Previous results

The papers (1), (2), (3) are reviewed below.

5.1. **Holomorphic Lagrangian branes and applications in representation theory.** In my thesis work (1), I built a natural link between holomorphic Lagrangian branes in the cotangent bundle of a complex manifold X and perverse sheaves on X. When X is the flag variety of a semisimple Lie group G, the category of perverse sheaves smooth along the Schubert cells are of particular interest in representation theory as they are equivalent to (a version of) the BGG category $\mathcal{O}$. In (3), I gave a construction of the holomorphic branes that represent the big tilting object in the BGG category $\mathcal{O}$.

5.2. **Symplectic symmetry groups.** In (2), I studied a version of G*-equivariant* symplectomorphism group of the cotangent bundle of flag varieties, denoted by $\mathrm{Sympl}^G(T^*(G/B))$, and proved that for $G = SL_n(\mathbb{C})$, there is a surjective group homomorphism from $\mathrm{Sympl}^G(T^*(G/B))$ to the braid group on n-strands B_{S_n} (which is an alternative construction than that in [SeSm]). In particular, I proved that for $G = SL_3(\mathbb{C})$, there is a homotopy equivalence between $\mathrm{Sympl}^G(T^*(G/B))$ and the braid group.

6. List of Papers

(1) Holomorphic Lagrangian branes correspond to perverse sheaves. *Geom. Topol.* 19 (2015), 1685–1735. https://arxiv.org/abs/1311.3756 https://doi.org/10.2140/gt.2015.19.1685

(2) Symplectomorphism group of $T^*(G/B)$ and the braid group I: A homotopy equivalence for $G = SL_3(\mathbb{C})$. *J. Symplectic Geom.* 17 (2) (2019), 337–380. https://arxiv.org/abs/1412.0511 https://dx.doi.org/10.4310/JSG.2019.v17.n2.a2

(3) Representing the big tilting sheaves as holomorphic Morse branes. *Adv. Math.* 345 (2019), 845–860. https://arxiv.org/abs/1602.07382 https://doi.org/10.1016/j.aim.2019.01.035

(4) Brane structures in microlocal sheaf theory. Joint with D. Treumann. *J. Topology* 17 (1) (2024), e12325, 68 pages. https://arxiv.org/abs/1704.04291 https://doi.org/10.1112/topo.12325

(5) A Hamiltonian $\coprod_n BO(n)$-action, stratified Morse theory and the J-homomorphism. *Compositio Math.* (to appear), 2024, 96 pages. https://arxiv.org/abs/1902.06708

(6) Hyperbolicity of asymmetric lemon billiards. Joint with P. Zhang, *Nonlinearity* 34 (1) (2021), 92–117. https://arxiv.org/abs/1902.08130 https://doi.org/10.1088/1361-6544/abaff2

(7) Microlocal sheaf categories and the J-homomorphism. *Submitted*, 55 pages. https://arxiv.org/abs/2004.14270

(8) Homological mirror symmetry for the universal centralizers I: the adjoint group case. (This is now part of item (11)). https://arxiv.org/abs/2107.13395

(9) Birkhoff normal form and twist coefficients of periodic orbits of billiards. Joint with P. Zhang. *Nonlinearity* 35 (8) (2022), 3907–3943. https://arxiv.org/abs/2108.12098 https://doi.org/10.1088/1361-6544/ac7701

(10) Homoclinic and heteroclinic intersections for lemon billiards. Joint with P. Zhang. *Adv. Math.* 442 (2024), 109588, 67 pages. https://arxiv.org/abs/2203.06477 https://doi.org/10.1016/j.aim.2024.109588

(11) Homological mirror symmetry for the universal centralizers. 119 pages. https://arxiv.org/abs/2206.09035

(12) Mirror symmetry for the affine Toda systems. Joint with Z. Yun. *In preparation.*

(13) Multiplicative universal centralizers: cluster structure and mirror symmetry. Joint with B. Webster. *In preparation.*

(14) Cohomology of the universal centralizers. *In preparation.*

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